

CONFORMAL LINEARIZATION VERSUS NONLINEARITY OF W -ALGEBRAS

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Abstract

We review the new approach to the theory of nonlinear W -algebras which is developed recently and called *conformal linearization*. In this approach W -algebras are embedded as subalgebras into some *linear conformal* algebras with a finite set of currents and most of their properties could be understood in a much simpler way by studying their linear counterpart. The general construction is illustrated by the examples of $u(N)$ -superconformal, $W(sl(N), sl(2))$, $W(sl(N), sl(N))$ as well as $W(sl(N), sl(3))$ algebras. Applications to the construction of realizations (included modulo null fields realizations) as well as central charge spectrum for minimal models of nonlinear algebras are discussed.

To appear in “Geometry and Integrable Models”, Eds.: P.N.Pyatov & S.N.Solodukhin,
World Scientific Publ. Co. (in press)

October 1995

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1 Introduction

Since the pioneer paper of Zamolodchikov [1], a lot of extended nonlinear conformal algebras (the W -type algebras) have been constructed and studied (see, e.g., [2] and references therein). The growing interest to this subject is motivated by many interesting applications of nonlinear algebras to the string theory, integrable systems, etc. However, the intrinsic nonlinearity of W -algebras makes it rather difficult to apply to them the standard arsenal of techniques and means used in the case of linear algebras (while constructing their field representations, etc.).

A way to circumvent this difficulty has been proposed by us in [3, 4, 5, 6]. We found that in many cases a given nonlinear W algebra can be embedded into some linear conformal algebra which is generated by a finite number of currents and contains the considered W -algebra as a subalgebra in some nonlinear basis. The currents of nonlinear algebra are related by an *invertible* transformation to those of linear one and most of the properties of the former and the theories constructed on its basis, can be understood in a more simple way by studying its linear counterpart. We called this linear algebra the linearizing algebra for the nonlinear one.

An idea to relate W -algebras and Lie algebras is also developed in [7], however in our approach there is an essential difference: our linearizing algebras are *conformal*, i.e. they contain Virasoro subalgebra and the remainder of their currents are *primary* with respect to Virasoro stress tensor. To underline this very important property of our linearizing procedure we call it *conformal linearization*. Up to now the explicit construction of conformal linearization has been carried out for many examples of nonlinear (super)algebras [3, 4, 5, 6, 8] and in all these cases conformal linearizing algebras are more *efficient*, i.e. contain less currents than the algebras of Ref. [7]. Besides being a useful tool to construct a new more wide class of field realizations of nonlinear algebras [3, 5], these linear conformal algebras provide a suitable framework for constructing new string theories as well as studying the embeddings of the Virasoro string in the W -type ones [9, 10, 11].

In the present review we would like to demonstrate that the conformal linearization is a general property inherent to many nonlinear W -algebras from the $W(sl(N), H)$ series produced via the Hamiltonian reduction constraints imposed on the affine $sl(N)$ currents, associated with principal embedding of $sl(2)$ algebra into regular subalgebra H of $sl(N)$ [2, 12, 13]. We describe the heuristic method of conformal linearization and present linearizing conformal algebras for a wide class of W -(super)algebras.

Our approach [4, 6] is based on the *Conjecture* about the relation between conformal linearizing algebra for $W(sl(N), H)$ and the linearizing algebra for the algebra \widetilde{W} obtained via special Hamiltonian reduction applied to conformal linearizing algebra for $W(sl(N), sl(2))$. The existence of correspondence between these linearizing algebras seems reasonable if one remembers that $W(sl(N), sl(2))$ algebra turns out to be more universal than $W(sl(N), H)$ in the sense that the latter can be generated via secondary Hamiltonian reductions from the former [14, 8]. The *Conjecture* as well as the method of construction of conformal linearizing algebras for $W(sl(N), sl(2))$ [4, 6] are inspired by the analysis of simplest examples of conformal linearization for W_3 ($W(sl(3), sl(3))$) and $W_3^{(2)}$ ($W(sl(3), sl(2))$) algebras [3]. After finding the explicit form of the conformal linearizing algebra for $W(sl(N), sl(2))$, we will show that its different Hamiltonian reductions can be linearized using a slight modification of the method used in [7] for the reductions of affine algebra $sl(N)$. However due to the fact that we started from reductions of conformal algebras, their linearizing algebras are also conformal.

We illustrate the general construction by the examples of $u(N)$ -superconformal [15], $W(sl(N), sl(2))$ [12, 16, 17], $W(sl(3), sl(3))$ [1], $W(sl(4), sl(4))$ [18, 19] as well as $W(sl(N), sl(3))$

algebras. The explicit formulas relating linearizing and nonlinear conformal algebras for all these cases are given and their new realizations are produced in this way.

An alternative approach to conformal linearization is developed independently in [8] in the framework of the quantum secondary Hamiltonian reduction. However the linearization of the $W(sl(N), sl(2))$ algebras was not considered in [8], because the method used there does not allow fields with negative conformal weights which are necessary for this purpose.

The review is organized as follows. In Section 2 we describe linearizing conformal algebras for $W(sl(N), sl(2))$ as well as for their supersymmetric counterpart - $u(N)$ superconformal algebras. In Section 3 we formulate the main *Conjecture* and in the framework of BRST formalism we find the general formulas for the currents of linearizing conformal algebras for $W(sl(N), H)$. In Sections 4 and 5 we apply general approach of Section 3, to $W(sl(N), sl(N))$ (W_N [18]) and $W(sl(N), sl(3))$ series of nonlinear algebras. And finally, in Section 6 we end with closing remarks.

2 Linearizing $W(sl(N + 2), sl(2))$ and $u(N)$ superconformal algebras.

In this Section we construct conformal linearizing algebras for $W(sl(N + 2), sl(2))$ [12, 16, 17] and $W(sl(N|2), sl(2))$ ($u(N)$ superconformal [15]) (super)algebras [4, 6].

Hereafter, some nonlinear redefinition of the currents is called the change of the basis of the (non)linear algebra, if (i) it is invertible and (ii) both it and its inverse are polynomial in the currents and derivatives of the latter. A subset of the currents is meant to form a (non)linear subalgebra of given W -algebra if in some basis this subset is closed; all the algebras related by (nonlinear) transformations of the basis are treated as equivalent.

Let us start by reminding the operator product expansions (OPEs) for the $W(sl(N + 2), sl(2))$ and $u(N)$ superconformal algebras (SCAs). The OPEs for these algebras can be written in a general uniform way keeping in mind that the $W(sl(N + 2), sl(2))$ algebra is none other than $u(N)$ quasi-superconformal algebra (QSCA) [12, 16, 17]¹. Both $u(N)$ SCA and $u(N)$ QSCA have the same number of generating currents: the stress tensor $T(z)$, the $u(1)$ current $U(x)$, the $su(N)$ affine currents $J_a^b(x)$ ($1 \leq a, b \leq N$, $\text{Tr}(J) = 0$) and two sets of currents in the fundamental $G_a(x)$ and conjugated $\bar{G}^b(x)$ representations of $su(N)$. The currents $G_a(x), \bar{G}^b(x)$ are bosonic for $u(N)$ QSCA and fermionic for $u(N)$ SCA. To distinguish between these two cases we, following Ref. [16], introduce the parameter ϵ equal to 1 (−1) for the QSCAs (SCAs) and write the OPEs for these algebras in the following universal form:

$$\begin{aligned} T(z_1)T(z_2) &= \frac{c/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}} \quad , \quad U(z_1)U(z_2) = \frac{c_1}{z_{12}^2} \quad , \\ T(z_1)J_a^b(z_2) &= \frac{J_a^b}{z_{12}^2} + \frac{J_a^{b'}}{z_{12}} \quad , \quad T(z_1)U(z_2) = \frac{U}{z_{12}^2} + \frac{U'}{z_{12}} \quad , \\ T(z_1)G_a(z_2) &= \frac{3/2G_a}{z_{12}^2} + \frac{G_a}{z_{12}} \quad , \quad T(z_1)\bar{G}^a(z_2) = \frac{3/2\bar{G}^a}{z_{12}^2} + \frac{\bar{G}^a}{z_{12}} \quad , \\ J_a^b(z_1)J_c^d(z_2) &= (K - \epsilon - N) \frac{\delta_a^d \delta_c^b - \frac{1}{N} \delta_a^b \delta_c^d}{z_{12}^2} + \frac{\delta_c^b J_a^d - \delta_a^d J_c^b}{z_{12}} \quad , \end{aligned}$$

¹Strictly speaking, the $W(sl(N + 2), sl(2))$ algebra coincides with $gl(N)$ QSCA. In what follows, we will not specify the real forms of algebras and use the common term $u(N)$ QSCA.

$$\begin{aligned}
U(z_1)G_a(z_2) &= \frac{G_a}{z_{12}} \quad , \quad U(z_1)\bar{G}^a(z_2) = -\frac{\bar{G}^a}{z_{12}} \quad , \\
J_a^b(z_1)G_c(z_2) &= \frac{\delta_c^b G_a - \frac{1}{N}\delta_a^b G_c}{z_{12}} \quad , \quad J_a^b(z_1)\bar{G}^c(z_2) = \frac{-\delta_a^c \bar{G}^b + \frac{1}{N}\delta_a^b \bar{G}^c}{z_{12}} \\
G_a(z_1)\bar{G}^b(z_2) &= \frac{2\delta_a^b c_2}{z_{12}^3} + \frac{2x_2\delta_a^b U + 2x_3 J_a^b}{z_{12}^2} + \frac{x_2\delta_a^b U' + x_3 J_a^{b'} + 2x_5(J_a^d J_d^b)}{z_{12}} \\
&\quad + \frac{2x_4(U J_a^b) + \delta_a^b(x_1(UU) - 2\epsilon T + 2x_6(J_d^e J_e^d))}{z_{12}} \quad ,
\end{aligned} \tag{2.1}$$

where the central charges c and parameters x are defined by

$$c = \frac{-6\epsilon K^2 + (N^2 + 11\epsilon N + 13)K - (\epsilon + N)(N^2 + 5\epsilon N + 6)}{K} \quad , \tag{2.2}$$

$$\begin{aligned}
c_1 &= \frac{N(2K - N - 2\epsilon)}{2 + \epsilon N} \quad , \quad c_2 = \frac{(K - N - \epsilon)(2K - N - 2\epsilon)}{K} \quad , \\
x_1 &= \frac{(\epsilon + N)(2\epsilon + N)}{N^2 K} \quad , \quad x_2 = \frac{(2\epsilon + N)(K - \epsilon - N)}{\epsilon N K} \quad , \quad x_3 = \frac{2K - N - 2\epsilon}{K} \quad , \\
x_4 &= \frac{2 + \epsilon N}{N K} \quad , \quad x_5 = \frac{1}{K} \quad , \quad x_6 = \frac{1}{2\epsilon K} \quad .
\end{aligned} \tag{2.3}$$

The currents in the r.h.s. of OPEs (2.1) are evaluated at the point z_2 , $z_{12} = z_1 - z_2$ and the normal ordering in the nonlinear terms is understood. Hereafter we will write only regular terms of OPEs.

The problem of construction of linear algebras for nonlinear ones can be naturally divided in two steps. At the first step we need to find the appropriate set of additional currents which linearize the given nonlinear algebra. In other words, we should construct the linear algebra for extended set of currents with the special relations between, for example, central charges, conformal weights etc., so that it contains the nonlinear algebra as a subalgebra in some nonlinear basis. At the second step, we need to explicitly construct the transformation from the linear basis to a nonlinear one. While the first step is highly non-trivial, the second one is purely technical: one writes down the most general expressions in the currents of linear algebra with arbitrary coefficients and conformal weights appropriate to nonlinear algebra currents, and then fixes all the coefficients from the OPEs of the given nonlinear algebra. In principle, if we know from some consideration the true linearizing algebra we do not need to know beforehand even OPEs of nonlinear algebra because all the information about nonlinear algebra is encoded in the linear one. We could derive OPEs of nonlinear algebra by demanding OPEs between the constructed general expressions to form a closed set. Below in this Section we consider these two steps for the algebras under consideration.

The main question we need to answer at first step in order to linearize the $u(N)$ (Q)SCA (2.1) is as to which minimal set of additional currents must be added to $u(N)$ (Q)SCA to get extended linear conformal algebras containing (2.1) as subalgebras in some nonlinear basis. The idea of our construction comes from the observation that the classical ($K \rightarrow \infty$) $u(N)$ (Q)SCA (2.1) can be realized as left shifts in the following coset space

$$g = e^{\int dz \bar{Q}^a(z) G_a(z)} \quad , \tag{2.4}$$

which is parametrized by N parameters-currents $\bar{Q}^a(z)$ with unusual conformal weights $-1/2$. In this case, the currents of $u(N)$ (Q)SCA (2.1) can be realized in terms of $\bar{Q}^a(z)$ and their conjugated momenta $G_a(z) = \delta/\delta\bar{Q}^a$ with OPEs

$$G_a(z_1)\bar{Q}^b(z_2) = \frac{\delta_a^b}{z_{12}}, \quad (2.5)$$

as well as the currents of the maximal *linear* subalgebra \mathcal{H}_N

$$\mathcal{H}_N = \{T, U, J_a^b, \bar{G}^a\}. \quad (2.6)$$

Moreover, a realization of a given current from $u(N)$ (Q)SCA will contain some linear term belonging to the set of currents $\{T, U, J_a^b, \bar{G}^a, \bar{Q}^a(z), G_a(z)\}$ which form the linear algebra. So actually such a realization describes the change of the basis from this linear algebra to the nonlinear algebra $u(N)$ (Q)SCA extended by the currents $\bar{Q}^a(z)$.

Though the situation in quantum case is more difficult, it still seems reasonable to try to generalize this classical picture to the quantum case, i.e. to extend the $u(N)$ (Q)SCA (2.1) by N additional currents $\bar{Q}^a(z)$ with conformal weights $-1/2$, especially keeping in mind that the current with just this conformal weight appears in the linearization of $W_3^{(2)}$ algebra [3].

Fortunately, this extension is sufficient to construct the linearizing algebra for the $u(N)$ (Q)SCA in the quantum case also. Without going into details, let us write down the set of OPEs for this linear algebra with the currents $\{T(z), U(z), J_a^b(z), G_a(z), \bar{G}^a(z), \bar{Q}^a(z)\}$, which we denote as $(Q)SCA_N^{lin}$ algebra

$$\begin{aligned} T(z_1)T(z_2) &= \frac{c/2}{z_{12}^4} + \frac{2T}{z_{12}^2} + \frac{T'}{z_{12}}, \quad U(z_1)U(z_2) = \frac{c_1}{z_{12}^2}, \\ T(z_1)J_a^b(z_2) &= \frac{J_a^b}{z_{12}^2} + \frac{J_a^{b'}}{z_{12}}, \quad T(z_1)U(z_2) = \frac{U}{z_{12}^2} + \frac{U'}{z_{12}}, \\ T(z_1)G_a(z_2) &= \frac{3/2G_a}{z_{12}^2} + \frac{G_a}{z_{12}}, \quad T(z_1)\bar{G}^a(z_2) = \frac{3/2\bar{G}^a}{z_{12}^2} + \frac{\bar{G}^a}{z_{12}}, \\ T(z_1)\bar{Q}^a(z_2) &= \frac{-1/2\bar{Q}^a}{z_{12}^2} + \frac{\bar{Q}^a}{z_{12}}, \\ J_a^b(z_1)J_c^d(z_2) &= (K - \epsilon - N) \frac{\delta_a^d \delta_c^b - \frac{1}{N} \delta_a^b \delta_c^d}{z_{12}^2} + \frac{\delta_c^b J_a^d - \delta_a^d J_c^b}{z_{12}}, \\ U(z_1)G_a(z_2) &= \frac{G_a}{z_{12}}, \quad U(z_1)\bar{G}^a(z_2) = -\frac{\bar{G}^a}{z_{12}}, \quad U(z_1)\bar{Q}^a(z_2) = -\frac{\bar{Q}^a}{z_{12}}, \\ J_a^b(z_1)G_c(z_2) &= \frac{\delta_c^b G_a - \frac{1}{N} \delta_a^b G_c}{z_{12}}, \quad J_a^b(z_1)\bar{G}^c(z_2) = \frac{-\delta_a^c \bar{G}^b + \frac{1}{N} \delta_a^b \bar{G}^c}{z_{12}}, \\ J_a^b(z_1)\bar{Q}^c(z_2) &= \frac{-\delta_a^c \bar{Q}^b + \frac{1}{N} \delta_a^b \bar{Q}^c}{z_{12}}, \\ G_a(z_1)\bar{Q}^b(z_2) &= \frac{\delta_a^b}{z_{12}}, \quad G_a(z_1)\bar{G}^b(z_2) = \text{regular}. \end{aligned} \quad (2.7)$$

Here the central charges c and c_1 are the same as in (2.3) and the currents $G_a(z), \bar{G}^a(z)$ and $\bar{Q}^a(z)$ are bosonic (fermionic) for $\epsilon = 1$ (-1).

At the second step, in order to prove that the linear algebra $(Q)SCA_N^{lin}$ (2.7) contains $u(N)$ (Q)SCA (2.1) as a subalgebra, let us perform the following *invertible* nonlinear transformation to the new basis $\{T(z), U(z), J_a^b(z), G_a(z), \bar{G}^a(z), \bar{Q}^a(z)\}$, where the "new" current $\bar{G}^a(z)$ is defined as

$$\begin{aligned}\bar{G}^a = & \tilde{\bar{G}}^a + y_1 \bar{Q}^{a''} + y_2 (J_b^a \bar{Q}^{b'}) + y_3 (U \bar{Q}^{a'}) + y_4 (J_b^{a'} \bar{Q}^b) + y_5 (U' \bar{Q}^a) + y_6 (T \bar{Q}^a) \\ & + y_7 (J_b^c J_c^a \bar{Q}^b) + y_8 (J_b^c J_c^b \bar{Q}^a) + y_9 (U J_b^a \bar{Q}^b) + y_{10} (U U \bar{Q}^a) + y_{11} (J_b^c G_c \bar{Q}^b \bar{Q}^a) \\ & + y_{12} (J_b^a G_c \bar{Q}^c \bar{Q}^b) + y_{13} (G_b' \bar{Q}^b \bar{Q}^a) + y_{14} (G_b \bar{Q}^{b'} \bar{Q}^a) + y_{15} (G_b \bar{Q}^b \bar{Q}^{a'}) \\ & + y_{16} (G_b G_c \bar{Q}^b \bar{Q}^c \bar{Q}^a) + y_{17} (U G_b \bar{Q}^b \bar{Q}^a) ,\end{aligned}\quad (2.8)$$

and the coefficients $y_1 - y_{17}$ are defined as

$$\begin{aligned}y_1 &= 2K , \quad y_2 = 4 , \quad y_3 = \frac{2(2 + \epsilon N)}{N} , \quad y_4 = \frac{2(K - \epsilon - N)}{K} , \\ y_5 &= \frac{(K - \epsilon - N)(2 + \epsilon N)}{NK} , \quad y_6 = -2\epsilon , \quad y_7 = \frac{2}{K} , \quad y_8 = \frac{2}{\epsilon K} , \quad y_9 = \frac{2(2 + \epsilon N)}{NK} , \\ y_{10} &= \frac{(\epsilon + N)(2\epsilon + N)}{N^2 K} , \quad y_{11} = y_{12} = \frac{2}{K} , \quad y_{13} = \frac{2(K - N - 2\epsilon)}{K} , \quad y_{14} = 4 \\ y_{15} &= 2 , \quad y_{16} = \frac{2}{\epsilon K} , \quad y_{17} = \frac{2(2 + \epsilon N)}{NK} .\end{aligned}\quad (2.9)$$

Now it is a matter of straightforward (though tedious) calculation to check that OPEs for the subset of currents $\{T(z), U(z), J_a^b(z), G_a(z)\}$ and $\bar{G}^a(z)$ (2.8) coincide with the basic OPEs of the $u(N)$ (Q)SCA (2.1).

Thus, we have shown that the linear algebra $(Q)SCA_N^{lin}$ (2.7) contains $u(N)$ (Q)SCA as a subalgebra in the nonlinear basis.

We close this Section with a few comments.

First of all, the pairs of currents $G_a(z)$ and $\bar{Q}^a(z)$ (with conformal weights equal to $3/2$ and $-1/2$, respectively) in (2.7) look like "ghost-anti-ghost" fields and so $(Q)SCA_N^{lin}$ algebra (2.7) can be simplified by means of the standard ghost decoupling transformations

$$\begin{aligned}U &= \tilde{U} - \epsilon(G_a \bar{Q}^a) , \\ J_a^b &= \tilde{J}_a^b - \epsilon(G_a \bar{Q}^b) + \delta_a^b \frac{\epsilon}{N} (G_c \bar{Q}^c) , \\ T &= \tilde{T} + \frac{1}{2} \epsilon (G_a' \bar{Q}^a) + \frac{3}{2} \epsilon (G_a \bar{Q}^{a'}) - \frac{\epsilon(2 + \epsilon N)}{2K} \tilde{U}' ,\end{aligned}\quad (2.10)$$

where the term with the derivative of the current \tilde{U} is added to ensure primarity of \tilde{U} with respect to the new stress tensor \tilde{T} . In this new basis the algebra $(Q)SCA_N^{lin}$ splits into the direct sum

$$(Q)SCA_N^{lin} = \Gamma_N \oplus (Q)\widetilde{SCA}_N^{lin} \quad (2.11)$$

of the ghost-anti-ghost algebra $\Gamma_N = \{\bar{Q}^a, G_b\}$ with the OPEs (2.5) and the algebra $(Q)\widetilde{SCA}_N^{lin} = \{\tilde{T}, \tilde{U}, \tilde{J}_a^b, \tilde{\bar{G}}^a\}$ with the following set of OPEs

$$\tilde{T}(z_1) \tilde{T}(z_2) = \frac{-6\epsilon K^2 + (N^2 + 13)K - (N^3 - N + 6\epsilon)}{2K z_{12}^4} + \frac{2\tilde{T}}{z_{12}^2} + \frac{\tilde{T}'}{z_{12}} ,$$

$$\begin{aligned}
\tilde{U}(z_1)\tilde{U}(z_2) &= \left(\frac{2NK}{2+\epsilon N}\right)\frac{1}{z_{12}^2}, \quad \tilde{T}(z_1)\tilde{J}_a^b(z_2) = \frac{\tilde{J}_a^b}{z_{12}^2} + \frac{\tilde{J}_a^{b'}}{z_{12}}, \\
\tilde{T}(z_1)\tilde{U}(z_2) &= \frac{\tilde{U}}{z_{12}^2} + \frac{\tilde{U}'}{z_{12}}, \\
\tilde{T}(z_1)\tilde{G}^a(z_2) &= \left(\frac{3}{2} + \frac{\epsilon(2+\epsilon N)}{2K}\right)\frac{\tilde{G}^a}{z_{12}^2} + \frac{\tilde{G}^a}{z_{12}}, \\
\tilde{J}_a^b(z_1)\tilde{J}_c^d(z_2) &= (K-N)\frac{\delta_a^d\delta_c^b - \frac{1}{N}\delta_a^b\delta_c^d}{z_{12}^2} + \frac{\delta_c^b\tilde{J}_a^d - \delta_a^d\tilde{J}_c^b}{z_{12}}, \\
\tilde{U}(z_1)\tilde{G}^a(z_2) &= -\frac{\tilde{G}^a}{z_{12}}, \quad \tilde{J}_a^b(z_1)\tilde{G}^c(z_2) = \frac{-\delta_a^c\tilde{G}^b + \frac{1}{N}\delta_a^b\tilde{G}^c}{z_{12}}, \\
\tilde{G}^a(z_1)\tilde{G}^b(z_2) &= \text{regular}, \tag{2.12}
\end{aligned}$$

Secondly, note that the linear algebra $(Q)\widetilde{SCA}_N^{lin}$ (2.12) has the same number of currents and the same structure relations as the maximal linear subalgebra \mathcal{H}_N (2.6) of $u(N)$ (Q)SCA (2.1), but with the "shifted" central charges and conformal weights. It is of importance that the central charges and conformal weights are strictly related as in (2.12)². Otherwise, with another relation between these parameters, we would never find the $u(N)$ (Q)SCA (2.1) in $(Q)\widetilde{SCA}_N^{lin}$. Thus, our starting assumption about the structure of linear algebra for $u(N)$ (Q)SCA coming from the classical coset realization approach, proved to be correct, modulo shifts of central charges and conformal weights.

Thirdly, let us remark that among the $u(N)$ (Q)SCAs there are many (super)algebras which are well known under other names. For example³:

$$\begin{aligned}
(Q)SCA(\epsilon = 1, N = 1) &\equiv W_3^{(2)} \quad \text{Ref. [20, 21]}, \\
(Q)SCA(\epsilon = -1, N = 1) &\equiv N = 2 \text{ SCA} \quad \text{Ref. [22]}, \\
(Q)SCA(\epsilon = -1, N = 2) &\equiv N = 4 \text{ SU}(2) \text{ SCA} \quad \text{Ref. [22]}.
\end{aligned}$$

Finally, the linear algebra $(Q)\widetilde{SCA}_N^{lin}$ (2.12) is homogeneous in the currents \tilde{G}^a , so they are null fields. Evidently we could consistently put them equal to zero, $\tilde{G}^a = 0$, and be left with the Miura realization (2.8) of the $u(N)$ (Q)SCA (2.1) in terms of currents \tilde{T}_{Vir} , \tilde{U} , \tilde{J}_a^b , \tilde{Q}^a and G_b , where we introduced decoupling basis in $(Q)\widetilde{SCA}_N^{lin}$ algebra with the new stress tensor \tilde{T}_{Vir}

$$\tilde{T}_{Vir} = \tilde{T} - \frac{1}{2K}\tilde{J}_a^b\tilde{J}_b^a - \frac{2+\epsilon N}{4NK}\tilde{U}\tilde{U}, \tag{2.13}$$

commuting with all other currents and having the following central charge c_{Vir}

$$c_{Vir} = 1 - 6\frac{(K-1)^2}{K}. \tag{2.14}$$

²Let us point out that Jacobi identities for the set of currents $\{\tilde{T}, \tilde{U}, \tilde{J}_a^b, \tilde{G}^a\}$ do not fix neither central charges nor the conformal weight of \tilde{G}^a .

³To avoid the singularity in (2.3) at $\epsilon = -1, N = 2$ one should firstly rescale the current $U \rightarrow \frac{1}{\sqrt{2+\epsilon N}}U$ and then put $\epsilon = -1, N = 2$ [15].

In this basis at $\widetilde{G}^a = 0$ the $(Q)\widetilde{SCA}_N^{lin}$ algebra (2.12) splits in a direct sum of Virasoro, $u(1)$ and $sl(N)$ affine algebras. The values of c_{Vir} corresponding to the minimal models of Virasoro algebra [23] at

$$K = \frac{p}{q} \Rightarrow c_{Vir} = 1 - 6 \frac{(p-q)^2}{pq} \quad (2.15)$$

induce the following spectrum for central charge c (2.2) of $u(N)$ (Q)SCA (2.1)

$$c = \frac{-6\epsilon p^2 + (N^2 + 11\epsilon N + 13)pq - (\epsilon + N)(N^2 + 5\epsilon N + 6)q^2}{pq}. \quad (2.16)$$

One can check for the particular cases of $N = 2$ superconformal and $W_3^{(2)}$ algebras that corresponding spectrum contains the spectrum of minimal models for these algebras [24, 21]. Moreover, as we will show in the Section 4 for the case of W_N algebra, spectrum of central charge for its minimal models can be reproduced by minimal models of this Virasoro algebra. So it seems reasonable to suppose that this property will remain also for the whole series of $u(N)$ (Q)SCAs. Nevertheless, our conjecture must be checked by the standard methods.

Besides these simplest realizations there are ones with null currents which are realized in terms of free fields by a non-vanishing operator. In Section 4.1 we will discuss such realizations for W_3 algebra.

In the simplest case of $W_3^{(2)}$ algebra, the linear $Q\widetilde{SCA}_1^{lin}$ algebra (2.12) coincides with the linear algebra W_3^{lin} [3] for W_3 . For general N the situation is more complicated. This question will be discussed in the next Section.

3 Secondary linearization of $W(sl(N+2), H)$ algebras.

In this Section following Refs. [4, 6] we demonstrate that the linear algebra $Q\widetilde{SCA}_N^{lin}$ (2.7) constructed in the previous Section gives the hints how to find the linearizing algebras for many other W -type algebras which can be obtained from the $W(sl(N+2), sl(2))$ ($u(N)$ QSCAs) via the secondary Hamiltonian reduction [14].

The $W(sl(N+2), sl(2))$ algebra, which have been linearized in the previous Section, can be obtained through the primary Hamiltonian reduction with *minimal* set of constraints from the affine $sl(N+2)$ algebras [12, 16, 17] and so at a fixed value of N it contains the *maximally* possible set of the currents. The full set of constraints on the currents of $sl(N+2)$ algebra which yield $W(sl(N+2), sl(2))$ read

$$\left(\begin{array}{cc|cccc} U & T & \overline{G}^1 & \overline{G}^2 & \dots & \overline{G}^N \\ 1 & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & G_1 & & & & \\ 0 & G_2 & & & & \\ \vdots & \vdots & & & & \\ 0 & G_N & & & & \end{array} \right). \quad (3.17)$$

The $W(sl(N+2), sl(2))$ algebras, forming in themselves a particular class of W -algebras with quadratic nonlinearity, are at the same time universal in the sense that a lot of other W -algebras with higher nonlinearity can be obtained from them via the secondary Hamiltonian reduction (e.g., W_N algebras, etc.) [14].

Let us consider a set of possible secondary Hamiltonian reductions of $W(sl(N+2), sl(2))$ algebra (3.17), (2.1) to nonlinear algebras which at the same time could be also produced via primary Hamiltonian reductions of affine $sl(N+2)$ algebra and belong to the $W(sl(N+2), H)$ series⁴. These are introduced by imposing the following constraints on the $W(sl(N+2), sl(2))$ currents

$$G_1 = 1 \quad , \quad G_2 = \dots = G_N = 0 \quad , \quad (3.18)$$

$$sl(N)|_{sl(2)} \quad , \quad (3.19)$$

where we denoted as $sl(N)|_{sl(2)}$ the set of constraints on the $sl(N)$ currents associated with an arbitrary embedding of $sl(2)$ algebra into $sl(N)$ subalgebra of $W(sl(N+2), sl(2))$.

The main *Conjecture* we will keep to in this Section is as follows

To find the linearizing algebra for a given nonlinear W algebra related to $W(sl(N+2), sl(2))$ through the Hamiltonian reduction constraints (3.18) and/or (3.19), one should apply the same reduction to its linearizing algebra $QSCA_N^{lin}$ (2.11) and then linearize the resulting algebra.

Taking into account that $QSCA_N^{lin}$ has the structure of direct sum (2.11) of the ghost–anti-ghost $\Gamma_N = \{\bar{Q}^a, G_b\}$ (2.5) and $(Q)\widetilde{SCA}_N^{lin} = \{\tilde{T}, \tilde{U}, \tilde{J}_a^b, \tilde{G}^a\}$ (2.12) algebras, as well as that the currents \bar{Q}^a are the gauge degree of freedom for the gauge transformations produced by constraints (3.18), one can get another, but equivalent following form of *Conjecture*

- i) To find the linearizing algebra for a given nonlinear W algebra related to $W(sl(N+2), sl(2))$ through the Hamiltonian reduction constraints (3.18) and (3.19), one should apply the reduction (3.19) to the linear algebra $Q\widetilde{SCA}_N^{lin}$ (2.12) and then linearize the resulting algebra.*
- ii) The algebra $Q\widetilde{SCA}_N^{lin}$ itself is the linearizing algebra for the reduction (3.18), i.e. for $W(sl(N+2), sl(3))$ algebra.*
- iii) Linearizing algebra for reduction (3.19) has the structure of direct sum of the algebra Γ_N and linearizing algebra for reductions (3.19) of algebra $Q\widetilde{SCA}_N^{lin}$ (2.12).*

Thus, in fact *Conjecture* reduces the problem of conformal linearization of the algebra W obtained from the nonlinear algebra $W(sl(N+2), sl(2))$ through the full set of the Hamiltonian reduction constraints (3.18) and/or (3.19) (i.e. W belongs to $W(sl(N+2), H)$ series) to the problem of linearization of the algebra \widetilde{W} obtained from the more simple *linear* algebra $Q\widetilde{SCA}_N^{lin}$ by imposing the relaxed set (3.19). At present, we are not aware of the rigorous proof of this assumption, but it works in many examples: W_3 , W_4 algebras (see Subsections 4.1 and 4.2) and in Section 5 we will prove the point ii) of *Conjecture* concerning $W(sl(N+2), sl(3))$ algebras.

Of course, the secondary Hamiltonian reduction (3.19), being applied to $QSCA_N^{lin}$, gives rise to a nonlinear algebra. However, the problem of its linearization as we will show below can be reduced to the linearization of reduction (3.19) applied to the affine subalgebra $sl(N) \subset Q\widetilde{SCA}_N^{lin}$,

⁴Let us remind that by definition the $W(sl(N), H)$ algebra is a nonlinear algebra produced via the primary Hamiltonian reduction constraints imposed on the affine $sl(N)$ currents, associated with principal embedding of $sl(2)$ algebra into regular subalgebra H of $sl(N)$, and this series forms the complete set of nonlinear algebras associated with different $sl(2)$ embeddings into $sl(N)$ [2, 12, 13].

which was already constructed in [7]. The resulting algebra will be just linearizing algebra for the nonlinear algebra we started with.

Let us briefly discuss the explicit construction of the linearizing algebra W^{lin} for the nonlinear algebra \widetilde{W} obtained from $Q\widetilde{SC}A_N^{lin}$ via the Hamiltonian reduction constraints (3.19).

Let \mathcal{J} be a current corresponding to the Cartan element t_0 of $sl(2)$ subalgebra. With respect to the adjoint action of t_0 the $sl(N)$ algebra can be decomposed into eigenspaces of t_0 with positive, null and negative eigenvalues h_a

$$sl(N) = (sl(N))_- \oplus (sl(N))_0 \oplus (sl(N))_+ \equiv \bigoplus_{h_a} (sl(N))_{h_a} . \quad (3.20)$$

(In this Section below, the latin indices (a, b) run over the whole $sl(N)$, Greek indices (α, β) run over $(sl(N))_-$ and the barred Greek ones $(\bar{\alpha}, \bar{\beta})$ over $(sl(N))_0 \oplus (sl(N))_+$.) The Hamiltonian reduction associated with the decomposition (3.20) can be performed by putting the appropriate constraints

$$J_\alpha - \chi_\alpha = 0 \quad , \quad \chi_\alpha \equiv \chi(J_\alpha) \quad (3.21)$$

on the currents J_α from $(sl(N))_-$ [2, 12]. These constraints are first class for integral gradings⁵, which means that BRST formalism can be used.

In order to impose the constraints (3.21) in the framework of BRST approach one can introduce the fermionic ghost–anti-ghost pairs (b_α, c^α) with ghost numbers $(-1, 1)$, respectively, for each current with the negative eigenvalues h_α , and with standard OPEs

$$c^\alpha(z_1)b_\beta(z_2) = \frac{\delta_\beta^\alpha}{z_{12}} \quad , \quad (3.22)$$

and the BRST charge

$$Q_{BRST} = \int dz J_{BRST}(z) = \int dz \left((J_\alpha - \chi(J_\alpha))c^\alpha - \frac{1}{2}f_{\alpha,\beta}^\gamma b_\gamma c^\alpha c^\beta \right) , \quad (3.23)$$

which coincides with that given in the paper [7]. The currents $\{\tilde{T}, \tilde{U}, \tilde{J}_a^b, \tilde{\bar{G}}^a\}$ of the algebra $Q\widetilde{SC}A_N^{lin}$ and the ghost fields $\{b_\alpha, c^\alpha\}$ form the BRST complex, graded by the ghost number. The \widetilde{W} algebra is defined in this approach as the algebra of operators generating the BRST charge null cohomology of this complex.

Following [7], let us introduce the "hatted" currents \hat{J}_a :

$$\hat{J}_a = \tilde{J}_a + \sum_{\beta,\gamma} f_{a,\beta}^\gamma b_\gamma c^\beta , \quad (3.24)$$

where $f_{a,\beta}^\gamma$ are structure constants of $sl(N)$ in the basis (3.20). As shown in [7], the W -algebras, associated with the reductions (3.19) of the affine $sl(N)$ can be embedded into linear algebras formed by the subset of non-constrained currents $\hat{J}_{\bar{\alpha}}$.

In contrast to the $sl(N)$ algebra, our algebra $Q\widetilde{SC}A_N^{lin}$ contains, besides the $sl(N)$ currents \tilde{J}_a^b , three additional ones $\tilde{T}, \tilde{U}, \tilde{\bar{G}}^a$. Fortunately, the presence of these currents create no new problems while we construct a linearizing algebra for the reduction of $Q\widetilde{SC}A_N^{lin}$ with the BRST

⁵Let us remind, that the half-integer gradings can be replaced by integer ones, leading to the same reduction [2].

charge (3.23). Namely, the improved stress-tensor \hat{T} with respect to which J_{BRST} in eq. (3.23) is a spin 1 primary current can be easily constructed

$$\hat{T} = \tilde{T} + \mathcal{J}' + \sum_{\alpha} \{-(1 + h_{\alpha})b_{\alpha}c^{\alpha'} - h_{\alpha}b'_{\alpha}c^{\alpha}\} , \quad (3.25)$$

so together with the zero ghost number current \tilde{U} it commutes with Q_{BRST} and belongs to both \tilde{W} and its linearizing algebra we are searching for. As regards the current \tilde{G}^i , one could check that it extends the complex generated by the currents $\hat{J}_a, b_{\alpha}, c^{\beta}$ with preserving the structure of the BRST subcomplexes of the paper [7], and forms, together with non-constrained currents $\hat{J}_{\bar{\alpha}}$ and c^{α} , a reduced BRST subcomplex and subalgebra which does not contain the currents with negative ghost numbers. Hence, following the same arguments which are given in [7]⁶, we can conclude that besides the non-constrained currents $\hat{J}_{\bar{\alpha}}$, the currents \tilde{G}^i also belong to the set of linearizing algebra currents for the \tilde{W} algebra and the last one closes not only modulo BRST exact operators, but in its own right.

Thus, the subset of the non-constrained currents $\hat{J}_{\bar{\alpha}}$ (3.24), stress tensor \hat{T} (3.25) and the currents

$$\hat{U} \equiv \tilde{U} \quad , \quad \hat{G} \equiv \tilde{G}^i \quad (3.26)$$

form the conformal linearizing algebra W^{lin} for the nonlinear algebra \tilde{W} . According to our main *Conjecture*, it forms, modulo algebra Γ_N , the linearizing algebra for W algebra obtained from $W(sl(N+2), sl(2))$ through the secondary Hamiltonian reduction associated with constraints (3.18) and/or (3.19).

Let us close this Section with a few remarks.

All results of this Section can be naturally generalized to the case of $W(sl(N|2), H)$ superalgebras. Thus all the considerations given here will be valid, if altogether $W(sl(N+2), H)$ algebras are replaced by $W(sl(N|2), H)$ superalgebras and in the sets of constraints (3.18) and (3.19) only (3.19) are kept (constraints (3.18) are forbidden for superalgebras due to fermionic statistic of current G_1), so in this case only the point iii) of *Conjecture* survives.

Conformal linearizing algebras constructed in this Section are more efficient compared to non-conformal linearizing algebras constructed in [7] for corresponding primary Hamiltonian reductions of $sl(N+2)$ affine algebra. Simple comparison shows that a given conformal linearizing algebra contains by $N+1$ (by 1) currents less than the one from [7] for the algebras corresponding to points i), ii) (to point iii)) of *Conjecture*.

In the next Sections we will illustrate the general construction of this Section by the examples of $W(sl(N), sl(N))$ (W_N) and $W(sl(N), sl(3))$ algebras.

4 Linearizing W_N algebras.

In this Section we apply the general procedure described in the previous Section 3 to the case of the principal embedding of $sl(2)$ into $sl(N)$ algebra to construct the linearizing algebras W_N^{lin} [4, 6] for W_N ($W(sl(N), sl(N))$) [18] algebras corresponding to the point i) of *Conjecture*.

For the principal embedding of $sl(2)$ into $sl(N)$ with the currents \hat{J}_a^b ($1 \leq a, b \leq N, Tr(J) = 0$), current \mathcal{J} which correspond to the Cartan element of $sl(2)$ is defined to be

⁶We don't reproduce here all details and refer a reader to the original paper [7].

$$\mathcal{J} = - \sum_{m=1}^{N-1} m \tilde{J}_{N-m}^{N-m} \quad , \quad (4.1)$$

and the decomposition of affine algebra $sl(N)$ reads as follows

$$\begin{aligned} (sl(N))_- &\propto \left\{ \tilde{J}_a^b, (2 \leq b \leq N, 1 \leq a < b) \right\}, \\ (sl(N))_0 \oplus (sl(N))_+ &\propto \left\{ \tilde{J}_a^b, (1 \leq a \leq N-1, a \leq b \leq N) \right\} \quad , \end{aligned} \quad (4.2)$$

i.e. $(sl(N))_-$ consists of those entries of the $N \times N$ current matrix which stand below the main diagonal, and the remainder just constitutes the subalgebra $(sl(N))_0 \oplus (sl(N))_+$.

Now, using general expressions of previous Section for the linearizing algebra currents (3.24) – (3.26) with principal gradation (4.1), (4.2) and OPEs (2.12), (3.22), and introducing the new stress tensor $\hat{\mathcal{T}}$

$$\hat{\mathcal{T}} = \hat{T} + \frac{(N+2)(K-1)}{2K} \hat{U}' \quad , \quad (4.3)$$

we are able to explicitly write OPEs for the currents of conformal linearizing algebra W_{N+2}^{lin} which contains the W_{N+2} algebra as a subalgebra:

$$\begin{aligned} \hat{\mathcal{T}}(z_1) \hat{\mathcal{T}}(z_2) &= \frac{(N+1) \left(1 - (N+2)(N+3) \frac{(K-1)^2}{K} \right)}{2z_{12}^4} + \frac{2\hat{\mathcal{T}}}{z_{12}^2} + \frac{\hat{\mathcal{T}}'}{z_{12}} \quad , \\ \hat{U}(z_1) \hat{U}(z_2) &= \left(\frac{2NK}{2+N} \right) \frac{1}{z_{12}^2} \quad , \\ \hat{\mathcal{T}}(z_1) \hat{J}_a^b(z_2) &= \frac{(N+1-2a)(K-1)\delta_a^b}{z_{12}^3} + \frac{(b-a+1)\hat{J}_a^b}{z_{12}^2} + \frac{\hat{J}_a^{b'}}{z_{12}} \quad , \\ \hat{\mathcal{T}}(z_1) \hat{U}(z_2) &= -\frac{2N(K-1)}{z_{12}^3} + \frac{\hat{U}}{z_{12}^2} + \frac{\hat{U}'}{z_{12}} \quad , \\ \hat{\mathcal{T}}(z_1) \hat{\bar{G}}^i(z_2) &= \frac{(i+2)\hat{\bar{G}}^i}{z_{12}^2} + \frac{\hat{\bar{G}}^{i'}}{z_{12}} \quad , \\ \hat{J}_a^b(z_1) \hat{J}_c^d(z_2) &= K \frac{\delta_a^d \delta_c^b - \frac{1}{N} \delta_a^b \delta_c^d}{z_{12}^2} + \frac{\delta_c^b \hat{J}_a^d - \delta_a^d \hat{J}_c^b}{z_{12}} \quad , \\ \hat{U}(z_1) \hat{\bar{G}}^i(z_2) &= -\frac{\hat{\bar{G}}^i}{z_{12}} \quad , \quad \hat{J}_a^b(z_1) \hat{\bar{G}}^i(z_2) = \frac{-\delta_a^i \hat{\bar{G}}^b + \frac{1}{N} \delta_a^b \hat{\bar{G}}^i}{z_{12}} \quad , \\ \hat{\bar{G}}^i(z_1) \hat{\bar{G}}^j(z_2) &= \text{regular} \quad , \end{aligned} \quad (4.4)$$

where the indices run over the following ranges:

$$\hat{J}_a^b : (1 \leq a \leq N-1, a \leq b \leq N) \quad , \quad \hat{\bar{G}}^i : (1 \leq i \leq N) \quad .$$

In this non-primary basis the currents $\hat{\bar{G}}^i$ have the same conformal weights $3, 4, \dots, N+2$ as the currents of W_{N+2} algebra, so the stress tensor $\hat{\mathcal{T}}$ coincides with the stress tensor of W_{N+2} algebra.

It is also instructive to rewrite the W_{N+2}^{lin} algebra (4.4) in the primary basis $\{\mathcal{T}, \hat{U}, \hat{J}_a^b, \hat{G}^i\}$, where a new stress-tensor \mathcal{T} is defined as

$$\mathcal{T} = \hat{\mathcal{T}} - \frac{(N+2)(K-1)}{2K} \hat{U}' + \frac{K-1}{K} \sum_{m=1}^{N-1} m \left(\hat{J}_{N-m}^{N-m} \right)' \quad (4.5)$$

and the OPEs have the following form

$$\begin{aligned} \mathcal{T}(z_1) \mathcal{T}(z_2) &= \frac{N+1-6\frac{(K-1)^2}{K}}{2z_{12}^4} + \frac{2\mathcal{T}}{z_{12}^2} + \frac{\mathcal{T}'}{z_{12}} \quad , \quad \hat{U}(z_1) \hat{U}(z_2) = \left(\frac{2NK}{2+N} \right) \frac{1}{z_{12}^2} \quad , \\ \mathcal{T}(z_1) \hat{J}_a^b(z_2) &= \frac{\left(1 - \frac{a-b}{K} \right) \hat{J}_a^b}{z_{12}^2} + \frac{\hat{J}_a^{b'}}{z_{12}} \quad , \\ \mathcal{T}(z_1) \hat{U}(z_2) &= \frac{\hat{U}}{z_{12}^2} + \frac{\hat{U}'}{z_{12}} \quad , \\ \mathcal{T}(z_1) \hat{G}^i(z_2) &= \frac{\left(\frac{3}{2} + \frac{1+2i}{2K} \right) \hat{G}^i}{z_{12}^2} + \frac{\hat{G}^{i'}}{z_{12}} \quad , \\ \hat{J}_a^b(z_1) \hat{J}_c^d(z_2) &= K \frac{\delta_a^d \delta_c^b - \frac{1}{N} \delta_a^b \delta_c^d}{z_{12}^2} + \frac{\delta_c^b \hat{J}_a^d - \delta_a^d \hat{J}_c^b}{z_{12}} \quad , \\ \hat{U}(z_1) \hat{G}^i(z_2) &= -\frac{\hat{G}^i}{z_{12}} \quad , \quad \hat{J}_a^b(z_1) \hat{G}^i(z_2) = \frac{-\delta_a^i \hat{G}^b + \frac{1}{N} \delta_a^b \hat{G}^i}{z_{12}} \quad , \\ \hat{G}^i(z_1) \hat{G}^j(z_2) &= \text{regular} \quad . \end{aligned} \quad (4.6)$$

In this basis the "chain" structure of linearizing algebras W_N^{lin} , i.e. the property of linearizing algebras with a given N to be subalgebras of those with a higher N , becomes most transparent. Namely, if we redefine the currents of W_{N+2}^{lin} as

$$\begin{aligned} \mathcal{U}_1 &= \hat{U} - N \sum_{m=1}^{N-1} \hat{J}_m^m \quad , \\ \mathcal{U} &= \frac{(N+2)(N-1)}{N(N+1)} \hat{U} + \frac{2}{N+1} \sum_{m=1}^{N-1} \hat{J}_m^m \\ \tilde{\mathcal{T}} &= \mathcal{T} + \sqrt{\frac{N+2}{12KN^2(N+1)}} \mathcal{U}_1 \quad , \quad \left(\text{or} \quad \tilde{\mathcal{T}} = \mathcal{T} - \frac{N+2}{2KN^2(N+1)} (\mathcal{U}_1 \mathcal{U}_1) \right) \quad , \\ \mathcal{J}_a^b &= \hat{J}_a^b - \frac{\delta_a^b}{N-1} \sum_{m=1}^{N-1} \hat{J}_m^m \quad , \quad (1 \leq a \leq N-2, a \leq b \leq N-1) \quad , \\ \mathcal{S}_a &= \hat{J}_a^N \quad , \quad (1 \leq a \leq N-1) \quad , \\ \bar{\mathcal{G}}^i &= \hat{G}^i \quad , \quad (1 \leq i \leq N-1) \quad , \\ \bar{\mathcal{Q}} &= \hat{G}^N \quad , \end{aligned} \quad (4.7)$$

then the subset $\tilde{\mathcal{T}}, \mathcal{U}, \mathcal{J}_a^b, \bar{\mathcal{G}}^i$ generates the algebra W_{N+1}^{lin} in the form (4.6). Thus, the W_{N+2}^{lin} algebras constructed have the following structure

$$W_{N+2}^{lin} = \{W_{N+1}^{lin}, \mathcal{U}_1, \mathcal{S}_a, \bar{\mathcal{Q}}\} \quad (4.8)$$

and therefore there exists the following chain of embeddings

$$\dots W_N^{lin} \subset W_{N+1}^{lin} \subset W_{N+2}^{lin} \dots \quad . \quad (4.9)$$

Let us stress that the nonlinear W_{N+2} algebras do not possess the chain structure like (4.9), this property is inherent only to their linearizing algebras W_{N+2}^{lin} .

By this we finished the construction of conformal linearizing algebras W_{N+2}^{lin} which contain W_{N+2} as subalgebras in a nonlinear basis. The explicit expression for the transformations from the currents of W_{N+2}^{lin} algebra to those forming W_{N+2} algebra is a matter of straightforward calculation once we know the exact structure of the linearizing algebra. In the next Subsections for the particular cases of W_3 and W_4 algebras we will consider this question in more detail.

Finally, let us stress that knowing the structure of the linearized algebras W_{N+2}^{lin} helps us to reveal some interesting properties of the W_{N+2} algebras and their representations.

First of all, each realization of W_{N+2}^{lin} algebra gives rise to a realization of W_{N+2} . Hence, the relation between linear and nonlinear algebras opens a way to find new non-standard realizations of W_{N+2} algebras. As was shown in [9, 10] for the particular case of W_3 , these new realizations (for details, see next Subsection) can be useful for constructions of new string theories and solving the problem of embedding Virasoro string into the W_3 one.

Among many interesting realizations of W_{N+2}^{lin} there is one very simple particular realization which can be described as follows. A careful inspection of the OPEs (4.6) shows that the currents

$$\widehat{\bar{G}}^i, \widehat{J}_a^b : (1 \leq a \leq N-1, a < b \leq N) \quad (4.10)$$

are null fields and so they can be consistently put equal to zero. In this case the algebra W_{N+2}^{lin} will contain only Virasoro stress tensor \mathcal{T} and N $u(1)$ -currents $\{\widehat{U}, \widehat{J}_1^1, \dots, \widehat{J}_{N-1}^{N-1}\}$. Of course, there exists the new decoupling basis, where all these currents commute with each other (see similar discussion at the end of Section 2). One can check that the currents of W_{N+2} algebra are realized in this basis in terms of some stress tensor T_{Vir} with the same central charge c_{Vir} as is given in eq. (2.14) and N decoupled commuting $u(1)$ currents. Surprisingly, the values of c_{Vir} corresponding to the minimal models of Virasoro algebra (2.15) [23] induce the central charge $c_{W_{N+2}}^{min.mod.}(p, q)$ of the minimal models for W_{N+2} algebra [18]

$$c_{W_{N+2}}^{min.mod.}(p, q) = (N+1) \left(1 - (N+2)(N+3) \frac{(p-q)^2}{pq} \right) \quad (4.11)$$

(let us remind that the stress tensor of W_{N+2} coincides with the stress tensor $\widehat{\mathcal{T}}$ in the non-primary basis (4.4)). For the W_3 algebra this property has been first discussed in [3].

4.1 Linearizing W_3 algebra.

In this Subsection, as an example of our construction, we present the explicit formulas concerning the conformal linearization of W_3 algebra [3].

The W_3 algebra [1] contains the currents $\{\widehat{\mathcal{T}}, \mathcal{W}\}$ with spins $\{2, 3\}$, respectively.

The structure of the linearizing algebra W_3^{lin} in the primary basis can be read off from the OPEs (4.6) by putting $N = 1$. So, the algebra W_3^{lin} contains the currents $\{\mathcal{T}, \widehat{U}, \widehat{\bar{G}}^1\}$, with the conformal weights $\{2, 1, \frac{3(K+1)}{2K}\}$, respectively.

Passing to the currents of W_3 goes over two steps.

Firstly, we must write down most general, nonlinear in the currents of W_3^{lin} , *invertible* expressions for the currents $\hat{\mathcal{T}}, \mathcal{W}$ with the desired conformal weights 2 and 3. It can be easily done in the nonprimary basis (4.4), where the stress tensor $\hat{\mathcal{T}}$ coincides with the stress tensor of W_3 algebra.

Secondly, we should calculate the OPEs between the constructed expressions and demand them to form a closed set.

This procedure completely fixes all the coefficients in the expressions for the currents of W_3 algebra in the primary basis in terms of currents of W_3^{lin} (up to unessential rescalings). Let us stress that we do not need to know the explicit structure of W_3 algebra. By performing the second step, we automatically reconstruct the W_3 algebra.

Let us present here the results of our calculations for the W_3 algebra

$$\begin{aligned}\hat{\mathcal{T}} &= \mathcal{T} + \frac{3(K-1)}{2K}\hat{U}', \\ \mathcal{W} &= \hat{G}^1 + \frac{6^{1/2}}{((5K-3)(5-3K))^{1/2}}(-(\hat{U}\mathcal{T}) + \frac{1}{K}(\hat{U}\hat{U}\hat{U}) + \\ &\quad \frac{3(K-1)}{2K}(\hat{U}\hat{U}') - \frac{K-1}{2}\mathcal{T}' + \frac{(K-1)^2}{4K}(\hat{U}'')) .\end{aligned}\quad (4.12)$$

Thus, all the remarkable nonlinear features of W_3 algebra can be traced to the choice of a nonlinear basis in the linear algebra W_3^{lin} . For example, every realization of W_3^{lin} is a realization of the W_3 algebra simultaneously⁷. So the problem of the construction W_3 -realizations is reduced to much more simple problem of constructing realizations of W_3^{lin} . In the rest of this Subsection we present an example of such a realizations.

From the simple structure of the W_3^{lin} algebra OPEs (4.6) at $N=1$ it is evident that its most general realization includes at least two free bosonic scalar fields ϕ_i ($i=1, 2$) with OPEs

$$\phi_i(z_1)\phi_j(z_2) = -\delta_{ij} \ln(z) , \quad (4.13)$$

as well as a commuting with them Virasoro stress tensor T_r having a nonzero central charge which we denote c_{T_r} . Representing the bosonic primary field \hat{G}^1 in the standard way by an exponential of ϕ_i , the current \hat{U} by the derivative of ϕ_i and \mathcal{T} by the sum of T_r and the standard stress-tensors of ϕ_i with background charges, and requiring them to satisfy the OPEs (4.6), we find the following expressions

$$\begin{aligned}\hat{G}^1 &= s \cdot \exp \left(i\sqrt{n - \frac{3}{2K}}\phi_2 + \frac{i\sqrt{3}}{\sqrt{2K}}\phi_1 \right) , \\ \hat{U} &= -i\sqrt{\frac{2K}{3}}\phi_1' , \\ \mathcal{T} &= T_r - \frac{1}{2}(\phi_1')^2 - \frac{1}{2}(\phi_2')^2 - \frac{i\left(3 - n + \frac{3}{2K}\right)}{2\sqrt{n - \frac{3}{2K}}}\phi_2'' , \\ c_{T_r} &= 3\frac{\left(3 - n + \frac{3}{K}\right)^2}{n - \frac{3}{2K}} - \frac{6(K-1)^2}{K} ,\end{aligned}\quad (4.14)$$

⁷Of course, the inverse statement is not correct in general.

where $n \in Z_+$ and s is an arbitrary parameter. Its arbitrariness reflects the invariance of the OPEs (4.6) with respect to rescaling of the null field \widehat{G}^1 . If $s \neq 0$, it can always be chosen, e. g., equal to unity by a constant shift of the field ϕ_1 .

In the case of $s = 0$ the obtained realizations can be simplified by introducing a new Virasoro stress-tensor \widetilde{T}_r with the central charge $c_{\widetilde{T}_r}$, which absorbs the field ϕ_2

$$\widetilde{T}_r = T_r - \frac{1}{2}(\phi_2')^2 - \frac{i\left(3 - n + \frac{3}{K}\right)}{2\sqrt{n - \frac{3}{2K}}}\phi_2'' , \quad (4.15)$$

$$c_{\widetilde{T}_r} = 1 - 6\frac{(K-1)^2}{K} \quad (4.16)$$

and ϕ_2 -dependence disappears altogether. In this notation the expressions (4.14) are given by

$$\begin{aligned} \widehat{G}^1 &= 0 , \\ \widehat{U} &= -i\sqrt{\frac{2K}{3}}\phi_1' , \\ \mathcal{T} &= \widetilde{T}_r - \frac{1}{2}(\phi_1')^2 . \end{aligned} \quad (4.17)$$

After substituting eqs. (4.14) into (4.12), we get a realization of the W_3 algebra which generalizes the realization obtained in [25] and is reduced to it at $s = 0$.

The discussed here realizations could be a starting point for constructing new versions of W_3 -string theories [9, 10]. For example, as was shown recently in [10] a special case of these realizations corresponding to $n = 0$, provides embedding of the Virasoro string into non-critical and critical W_3 strings.

4.2 Linearizing W_4 algebra.

In this Subsection, we continue our consideration of algebras belonging to W_N series, and discuss the conformal linearization of W_4 algebra [4, 6, 8].

The W_4 algebra [19] contains the currents $\{\widehat{\mathcal{T}}, \mathcal{W}, \mathcal{V}\}$ with spins $\{2, 3, 4\}$, respectively.

The structure of the linearizing algebra W_4^{lin} in the primary basis can be read off from the OPEs (4.6) by putting $N = 2$. So, the algebra W_4^{lin} contains the currents $\{\mathcal{T}, \widehat{U}, \widehat{\mathcal{J}}_1^1, \widehat{\mathcal{J}}_1^2, \widehat{G}^1, \widehat{G}^2\}$, with the conformal weights $\{2, 1, 1, \frac{K+1}{K}, \frac{3(K+1)}{2K}, \frac{3K+5}{2K}\}$, respectively.

Using exactly the same arguments as given in previous Subsection, we write down most general, nonlinear in the currents of W_4^{lin} , *invertible* expressions for the currents $\widehat{\mathcal{T}}, \mathcal{W}, \mathcal{V}$ with the desired conformal weights 2, 3 and 4. It can be easily done in the nonprimary basis (4.4), where the stress tensor $\widehat{\mathcal{T}}$ coincides with the stress tensor of W_4 algebra. After that we calculate the OPEs between the constructed expressions and demand them to form a closed set.

The results of our calculations for the W_4 algebra look as follows

$$\begin{aligned} \widehat{\mathcal{T}} &= \mathcal{T} + \frac{2(K-1)}{K}\widehat{U}' - \frac{K-1}{K}\widehat{\mathcal{J}}_1^1 , \\ \mathcal{W} &= \widehat{G}^1 + \frac{K-1}{K}(T_1 - T_2)' + \frac{1}{K}\left((T_1 - T_2)\widehat{U}\right) - \frac{K-1}{K}\widehat{\mathcal{J}}_1^2 - \frac{1}{K}(\widehat{\mathcal{J}}_1^1\widehat{U}) , \end{aligned}$$

$$\begin{aligned}
\mathcal{V} = & -\widehat{G}^2 + \frac{K-1}{K}\widehat{G}^{1'} + \frac{1}{2K}((\widehat{J}_1^2\widehat{J}_1^2) + \widehat{J}_1^{2''}) + \frac{1}{K}((\widehat{U} - 2\widehat{J}_1^1)\widehat{G}^1) - \frac{1}{K}((T_1 - T_2)\widehat{J}_1^2) \\
& + \frac{1}{2K}((T_1 - T_2)(T_1 - T_2)) - \frac{2}{K^2}(\widehat{J}_1^1\widehat{J}_1^2)' + \frac{1}{K^2}((T_1 + T_2)(2(K-1)\widehat{U}' + (\widehat{U}\widehat{U}))) \\
& + \frac{K-1}{K^2}((T_1 + T_2)'\widehat{U}) + \frac{(K-1)^2}{2K^2}(T_1 + T_2)'' + \frac{(K-1)}{3K^2}\widehat{U}''' \\
& + \frac{(2-K)(2K-1)}{4K^3}(\widehat{U}''\widehat{U}) + \frac{(K-1)^2}{K^3}(\widehat{U}'\widehat{U}') + \frac{16(3-2K)(3K-2)}{K(300-637K+300K^2)}(\mathcal{T}\mathcal{T}) \\
& + \frac{-120K^4 + 430K^3 - 617K^2 + 430K - 120}{4K^2(300-637K+300K^2)}\mathcal{T}'' ,
\end{aligned} \tag{4.18}$$

where the auxiliary currents T_1 and T_2 are defined as

$$\begin{aligned}
T_1 &= \mathcal{T} - \frac{1}{K}(\widehat{J}_1^1\widehat{J}_1^1) - \frac{1}{2K}(\widehat{U}\widehat{U}) , \\
T_2 &= \frac{1}{K}(\widehat{J}_1^1\widehat{J}_1^1) - \frac{K-1}{K}\widehat{J}_1^{1'} .
\end{aligned} \tag{4.19}$$

Let us repeat again that from the beginning we do not need to know the explicit structure of W_4 algebra except for the spin content of its currents. By performing the calculations we automatically reconstruct the W_4 algebra.

For the W_4^{lin} algebra (4.6) the currents $\widehat{G}^1, \widehat{G}^2$ and \widehat{J}_1^2 are null-fields. So we can consistently put them equal to zero. In this case the expressions (4.18) provide us with the Miura realization of W_4 algebra in terms of two currents with conformal spins 2 (T_1, T_2) and with the same central charges, and one current with spin 1 (\widehat{U}) which commute with each other.

5 Linearizing $W(sl(N+2), sl(3))$ algebras.

In this Section following Refs. [6, 27] we prove the point ii) of the main *Conjecture* of Section 3 concerning linearizing algebras for $W(sl(N+2), sl(3))$ by explicit construction of invertible expressions for the currents of $W(sl(N+2), sl(3))$ algebra in terms of currents of $Q\widetilde{SC}A_N^{lin}$ (2.12).

Let us briefly remind the conformal spin content of $W(sl(N+2), sl(3))$ algebra [12]. It consists of the following currents: spin-two stress tensor \mathcal{T} , spin-one $sl(N-1)$ ($N > 1$) and $u(1)$ affine currents \mathcal{J}_a^b and \mathcal{U} , respectively, commuting with them spin-three current \mathcal{W} , and two multiplets of spin-two currents \mathcal{T}_a and $\overline{\mathcal{T}}^a$ having opposite $u(1)$ charge and belonging to the fundamental and its conjugated representations of $sl(N-1)$.

Using exactly the same approach as given in Subsections 4.1 and 4.2 for the cases of W_3 and W_4 algebras, we introduce nonprimary basis for the currents of $Q\widetilde{SC}A_N^{lin}$ algebra (2.12) with new stress tensor

$$\widehat{\mathcal{T}} = \widetilde{\mathcal{T}} + \frac{(N+2)(K-N)}{2NK}\widetilde{\mathcal{U}}' + \widetilde{\mathcal{J}}_1^{1'} , \tag{5.20}$$

with the following central charge $c_{W(sl(N+2), sl(3))}$

$$c_{W(sl(N+2), sl(3))} = N^2 + 24N + 25 - 24K - \frac{N^3 + 6N^2 + 11N + 6}{K} . \tag{5.21}$$

In this basis the currents $\{\hat{\mathcal{T}}, \tilde{U}, \tilde{J}_1^1, \tilde{J}_a^b, \tilde{J}_a^1, \tilde{J}_1^a, \tilde{G}^1, \tilde{G}^a\}$ (where here $2 \leq a, b \leq N$) have the spins $\{2, 1, 1, 1, 2, 0, 3, 2\}$, respectively, so stress tensor $\hat{\mathcal{T}}$ coincides with the stress tensor of the $W(sl(N+2), sl(3))$ algebra. In this basis we write down most general, nonlinear in the currents of $Q\widehat{SCA}_N^{lin}$ algebra, *invertible* expressions for the currents $\{\mathcal{W}, \mathcal{J}_a^b, \mathcal{U}, \mathcal{T}_a, \overline{\mathcal{T}}^a\}$ with the desired conformal weights $\{3, 1, 1, 2, 2\}$, calculate the OPEs between the constructed expressions and demand them to form a closed set.

Let us present here the results of our calculations

$$\begin{aligned}
\mathcal{J}_a^b &= \tilde{J}_a^b + \frac{1}{N-1} \tilde{J}_1^1 \delta_a^b, \\
\mathcal{U} &= -\frac{2(N-1)}{N} \tilde{U} + 2\tilde{J}_1^1, \\
\mathcal{T}_a &= \tilde{J}_a^1, \\
\overline{\mathcal{T}}^a &= \tilde{G}^a + \frac{1}{3K(K-N)} \left(-\frac{2K^2-3K+2}{2} \tilde{J}_1^{a''} - (K-N-1)(\mathcal{J}_b^{a'} \tilde{J}_1^b) - (2K-1)(\mathcal{J}_b^a \tilde{J}_1^{b'}) \right. \\
&\quad - \frac{(N+2)(K-1)}{2N} (\tilde{U}' \tilde{J}_1^a) - \frac{(N+2)(K-1)}{N} (\tilde{U} \tilde{J}_1^{a'}) + \frac{2K-N^2-N-2}{2(N-1)} (\tilde{J}_1^{1'} \tilde{J}_1^a) \\
&\quad + \frac{2K+N-2}{N-1} (\tilde{J}_1^1 \tilde{J}_1^{a'}) - \frac{1}{2} (\mathcal{J}_b^c \mathcal{J}_c^b \tilde{J}_1^a) - (\mathcal{J}_b^c \mathcal{J}_c^a \tilde{J}_1^b) - \frac{N+2}{N} (\tilde{U} \mathcal{J}_b^a \tilde{J}_1^b) \\
&\quad + \frac{2}{N-1} (\mathcal{J}_b^a \tilde{J}_1^1 \tilde{J}_1^b) - \frac{N^2-N+2}{2(N-1)^2} (\tilde{J}_1^1 \tilde{J}_1^1 \tilde{J}_1^a) + \frac{N+2}{N(N-1)} (\tilde{U} \tilde{J}_1^1 \tilde{J}_1^a) \\
&\quad \left. - \frac{(N+1)(N+2)}{2N^2} (\tilde{U} \tilde{U} \tilde{J}_1^a) + K(\tilde{T} \tilde{J}_1^a) - (\tilde{J}_b^1 \tilde{J}_1^a \tilde{J}_1^b) \right), \tag{5.22}
\end{aligned}$$

$$\begin{aligned}
\mathcal{W} &= -\tilde{G}^1 + \frac{1}{3(N-3K+2)} \left(\frac{N+2}{N} (\tilde{T} \tilde{U}) + \frac{2K}{(K-N)} (\tilde{T} \tilde{J}_1^1) - \frac{N-3K+2}{2} \tilde{T}' \right. \\
&\quad + \frac{(N+2)^2(K+5KN-N^2-3N-2)}{6KN^3(N-3K+2)} (\tilde{U} \tilde{U} \tilde{U}) + \frac{(N+2)(K+2KN-N-2)}{N^2(K-N)(N-3K+2)} (\tilde{U} \tilde{U} \tilde{J}_1^1) \\
&\quad - \frac{(N+2)(K-N-2)(2K-5KN+N^2+2N)}{2NK(K-N)(N-1)(N-3K+2)} (\tilde{U} \tilde{J}_1^1 \tilde{J}_1^1) \\
&\quad + \frac{(K^2(11N^2-13N+2) - K(6N^3+N^2-20N+4) + N^4+2N^3-4N^2-8N)}{3K(K-N)(N-1)^2(N-3K+2)} (\tilde{J}_1^1 \tilde{J}_1^1 \tilde{J}_1^1) \\
&\quad + \frac{(N-2)(N-3K+2)}{K(N-1)(K-N)} (\tilde{J}_1^1 \tilde{J}_a^1 \tilde{J}_1^a) + \frac{(N+2)(N-3K+2)}{KN(K-N)} (\tilde{U} \tilde{J}_a^1 \tilde{J}_1^a) \\
&\quad - \frac{(N+2)(K-N-2)}{2KN(K-N)} (\tilde{U} \mathcal{J}_a^b \mathcal{J}_b^a) - \frac{KN-K+N+2}{K(N-1)(K-N)} (\mathcal{J}_a^b \mathcal{J}_b^a \tilde{J}_1^1) \\
&\quad + \frac{(N-3K+2)}{K(K-N)} (\mathcal{J}_a^b \tilde{J}_b^1 \tilde{J}_1^a) - \frac{N-3K+2}{3K(K-N)(K-1)} (\mathcal{J}_a^b \mathcal{J}_b^c \mathcal{J}_c^a) \\
&\quad - \frac{(N+2)(K^2+4KN-2K-N^2-2N)}{2NK(K-N)} (\tilde{U} \tilde{J}_1^{1'}) - \frac{(N+2)(K+N-2)}{2N(K-N)} (\tilde{U}' \tilde{J}_1^1) \\
&\quad \left. + \frac{N-3K+2}{K} (\tilde{J}_a^{1'} \tilde{J}_1^a) - \frac{K^2(5N-2) - K(3N^2+16N-4) + 4N^2+8N}{2K(K-N)(N-1)} (\tilde{J}_1^{1'} \tilde{J}_1^1) \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{(3K^2 - 3NK - 12K + 4N + 8)(N - 3K + 2)}{6K(K - N)(K - 1)}(\mathcal{J}_a^{b'}\mathcal{J}_b^a) + \frac{2(K - 1)(N - 3K + 2)}{K(K - N)}(\tilde{J}_a^1\tilde{J}_1^{a'}) \\
& - \frac{(N + 2)(K + 2NK - N - 2)}{2N^2K}(\tilde{U}'\tilde{U}) - \frac{(N + 2)(3K^2 + 2NK - 2K - N^2 + 4)}{12NK}(\tilde{U}'') \\
& - \frac{3K^3 + K^2(17N - 2) - K(9N^2 + 22N - 4) + N^3 + 6N^2 + 8N}{6K(K - N)}(\tilde{J}_1^{1''}), \tag{5.23}
\end{aligned}$$

where here the indices a, b, \dots run over the following ranges: $2 \leq a, b \leq N$.

Let us remind that for the $(Q)\widetilde{SCA}_N^{lin}$ algebra (2.12) the currents \widetilde{G}^a are null fields and we can consistently put them equal to zero (see discussion at the end of Section 2). In this case the expressions (5.20), (5.22), (5.23) provide us with the Miura realization of $W(sl(N + 2), sl(3))$ algebra in terms of currents $\tilde{T}, \tilde{U}, \tilde{J}_a^b$. Using the arguments similar to those in the end of the Section 2 we have following conjecture for the spectrum of central charges of $W(sl(N + 2), sl(3))$ algebra minimal models

$$c_{W(sl(N+2), sl(3))} = N^2 + 24N + 25 - \frac{24p^2 + (N^3 + 6N^2 + 11N + 6)q^2}{pq}. \tag{5.24}$$

At the end of this Section we would like to briefly discuss one more application of formulas (5.20), (5.22), (5.23) to the construction of the so called modulo null fields realizations for W_3 algebra. In contradistinction to ordinary realizations, for such ones in the OPE of spin-three current \mathcal{W} with itself, besides the standard terms, some nonzero spin-four operator \mathcal{V} is also present⁸

$$\mathcal{W}(z_1)\mathcal{W}(z_2) = \text{standard terms} + \frac{\mathcal{V}}{z_{12}} + \frac{\mathcal{V}'}{2z_{12}}. \tag{5.25}$$

There is one strong restriction on this operator: its OPE with itself must contain no central term, i.e.

$$< \mathcal{V}\mathcal{V} > = 0, \tag{5.26}$$

nevertheless in the r.h.s. of this OPE another current possessing the same property (5.26) could appears. All such currents are called null fields and form the ideal of the algebra. Due to the last property they can be consistently set equal to zero and on the shell of these constraints spin 2 and 3 currents form a realization of W_3 algebra.

From the above definition it is clear that the problem of construction of the realizations modulo null fields is very complicated one. However it can be reduced to an easier task of construction of the ordinary realizations, but for bigger algebras, containing more currents than W_3 and including the OPE (5.25) among the full set of its OPEs. Then for some discrete values of the central charge, null field condition (5.26) for the spin-four operator of such algebra could be satisfied and so at this particular values of the central charge the realizations of such bigger algebra form simultaneously a W_3 realization modulo null fields.

As has been shown in the beginning of this Section, the $W(sl(N + 2), sl(3))$ algebras contain the currents with spins 2 and 3 and so on their basis we can apply the above mentioned approach for construction of modulo null fields realizations of W_3 algebra.

⁸This operator could be composite or elementary.

Following [26, 27] we introduce the new stress tensor \mathcal{T}_w

$$\mathcal{T}_w = \hat{\mathcal{T}} - \frac{1}{2(K-1)} \mathcal{J}_a^b \mathcal{J}_b^a - \frac{N+2}{8(N-1)(3K-N-2)} \mathcal{U}\mathcal{U}, \quad (5.27)$$

with the central charge c_{W_3}

$$c_{W_3} = -\frac{(4K-N-2)(3K-N-3)(2K-N-1)}{K(K-1)}, \quad (5.28)$$

which together with the spin-three current \mathcal{W} commutes with the currents \mathcal{U} , \mathcal{J}_a^b and belongs to the coset

$$\frac{W(sl(N+2), sl(3))}{u(1) \oplus sl(N-1)}. \quad (5.29)$$

Substituting (5.20), (5.22) into (5.27), (5.23) and using the basic OPEs (2.12), one can check that OPE of spin 3 current with itself looks like (5.25) and spin-four operator⁹ becomes the null field at the following values of central charge c_{W_3} and corresponding values of parameter K [26, 27]

$$\begin{aligned} c_{W_3} = c_{W_3}^{min.mod.}(3, 2) = -2 &\Rightarrow K = \frac{N+2}{2}, (N \neq 2); K = \frac{N+1}{3}, (N \neq 2); K = \frac{N+3}{4}; \\ c_{W_3} = c_{W_3}^{min.mod.}(5+N, 2+N) &= \frac{2(N-7)(N+14)}{(N+2)(N+5)} \Rightarrow K = \frac{N+5}{3}, (N \neq 7); \end{aligned} \quad (5.30)$$

$$c_{W_3} = c_{W_3}^{min.mod.}(4-N, 6) = \frac{2(N+4)(2N+1)}{N-4} \Rightarrow K = \frac{N+2}{6}, (N \neq 4); \quad (5.31)$$

where $c_{W_3}^{min.mod.}(p, q)$ is defined by eq. (4.11) at $N = 1$. So just for these values of parameter K every realization of \widetilde{QSCA}_N^{lin} algebra (2.12) induces modulo null fields realization of W_3 algebra with the currents \mathcal{T}_w (5.27) and \mathcal{W} (5.23). A first attempt to classify the possible algebras which allow a contraction to W_N is made in [28] (see also references therein) where the central charge spectrum (5.30) was conjectured. The central charge spectrum (5.31) have been constructed only very recently in [27].

6 Conclusion.

In this review we have described the class of linear (super)conformal algebras with finite numbers of generating currents which contain in some nonlinear basis a wide class of W -(super)algebras, including $W(sl(N+2), sl(2))$, $W(sl(N|2), sl(2))$ ($u(N)$ -superconformal), $W(sl(N+2), sl(3))$ as well as W_N nonlinear algebras. The discussed algebras do not exhaust all examples of conformal linearization and we refer the interested reader to the original papers [5, 26] where $W(sl(3|1), sl(3))$, $W_{2,4}$ [29, 19], WB_2 [30] and spin 5/2 [1] (super)algebras are analyzed. We have illustrated some applications of conformal linearization. Thus using the relations between linearizing and nonlinear algebras we predicted the spectrum of central charges for $u(N)$ (Q)SCA and

⁹For considered algebras spin-four operator \mathcal{V} is very complicated composite operator and we do not reproduce here its explicit expression.

$W(sl(N), sl(3))$ minimal models as well as constructed large class of their realizations, including the induced modulo null fields realizations for W_3 algebra.

We do not have a rigorous proof of our main *Conjecture* of Section 3, but we have shown that it works for a wide class of nonlinear W -algebras corresponding to the points i) and ii) of *Conjecture*. It is very interesting to extend this investigation to the technically more complicated case of algebras associated with the point iii)¹⁰. The explicit construction of the linearizing algebras W_{N+2}^{lin} for W_{N+2} reveals many interesting properties of these algebras: they have a "chain" structure (i.e. the linear algebras with a given N are subalgebras of those with a higher N), the central charge of the Virasoro subsector of these linear algebras in the parametrization corresponding to the Virasoro minimal models, while putting the null fields equal to zero, induces the central charge for the minimal models of W_N , etc. These are some of the reasons why we believe that our conjecture is true. Nevertheless, it would be very interesting either to fully confirm the *Conjecture* from some first principles or to find the restrictions on the range of its applicability.

Method of conformal linearization for $W(sl(N+2), H)$ ($W(sl(N|2), H)$) algebras described here admits a natural generalization to a larger class of nonlinear superalgebras $W(sl(N, M), H)$. This work is in progress now.

We have explicitly demonstrated in the case of W_3 , W_4 and $W(sl(N+2), sl(3))$ algebras that we do not need to know beforehand the structure relations of the nonlinear algebras, which rapidly become very complicated with growth of spins of the involved currents. Once we have constructed the linearizing algebra, we could reproduce the structure of the corresponding nonlinear one. So, one of the open questions now is how much information about the properties of a given nonlinear algebra we can extract from its linearizing algebra. The answer to this question could be important for applications of linearizing algebras to W -strings, integrable systems with W -type symmetry, etc..

Acknowledgments.

It is a pleasure for us to thank S. Bellucci, A. Honecker, E. Ivanov and V. Ogievetsky for many interesting and clarifying discussions. One of us (A.S.) is also indebted to G. Zinovjev for his interest in this work and useful discussions.

This investigation has been supported in part by the Russian Foundation of Fundamental Research, grant 93-02-03821, and the International Science Foundation, grant M9T000.

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¹⁰For this type algebra among the currents of linearizing algebra (in the basis where stress tensor coincides with stress tensor of nonlinear one) there are present currents with negative conformal spins, and chargeless spin-zero composite operators be constructed. So the transformation to the basis where linearizing algebra contains nonlinear one as subalgebra, in principle could contain an infinite number of terms including these operators.

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